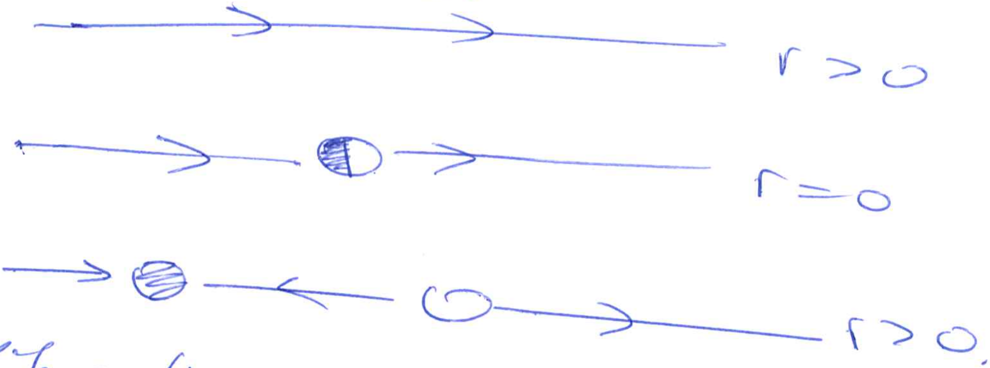
Bifurcations

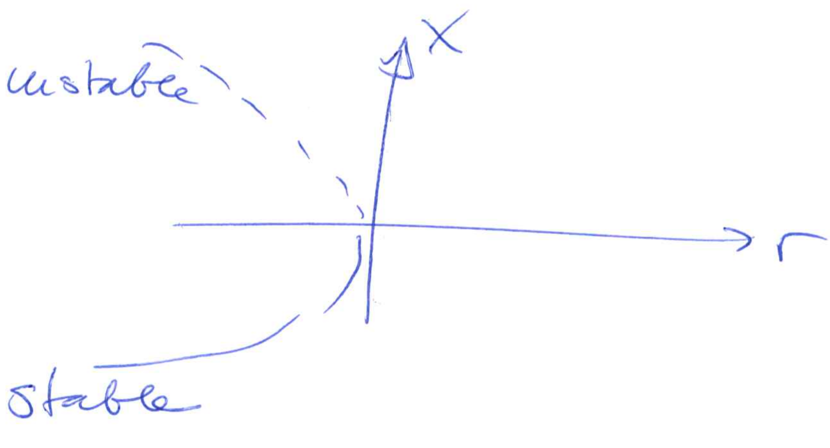
= Topology of phase space flow changes as control parameter is varied

<u>Example:</u> $\dot{x} = r - x^2$.

flow on a line:



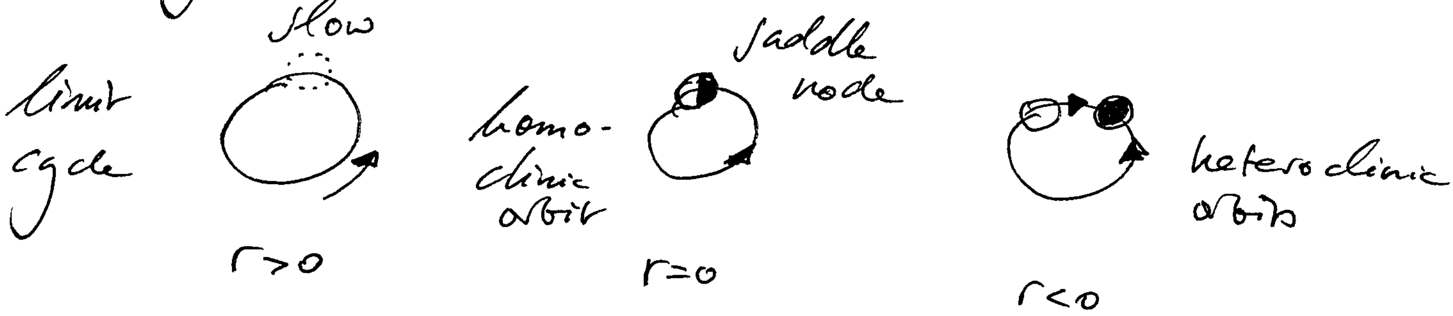
bifurcation diagram



Saddle-node-bifurcation:

two-fixed-points collide + annihilate each other.

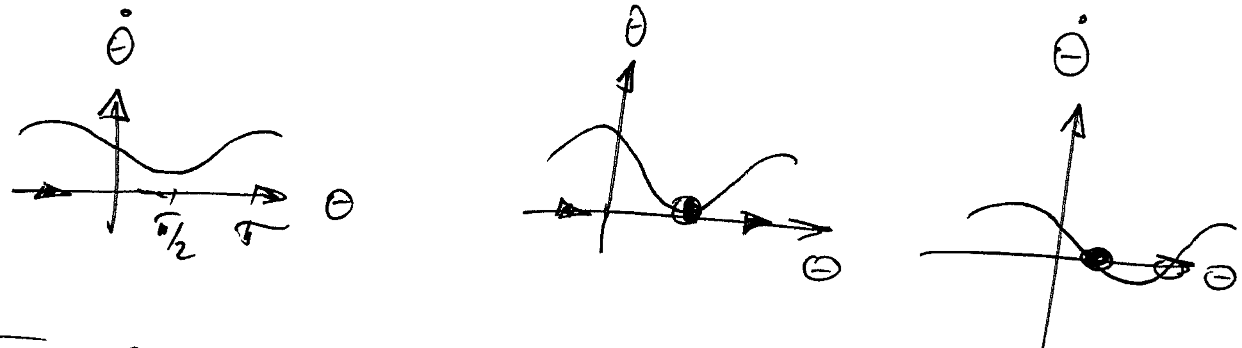
Ghost dynamics of saddle-node-bifurcation



bifurcation   oscillator stalled

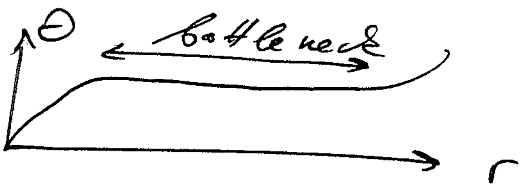
$$\dot{\theta} = \omega_0 - a \sin \theta .$$

- electronics:
  (phase-locked loops)
- biology
  (oscillating neurons, fire flies, ...)
- condensed matter physics
  (Josephson junctions



$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int^{2\pi} \frac{d\theta}{\omega - a \sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

$\sim \frac{1}{\sqrt{r}}$, $r = \omega - a$ bifurcation parameter.

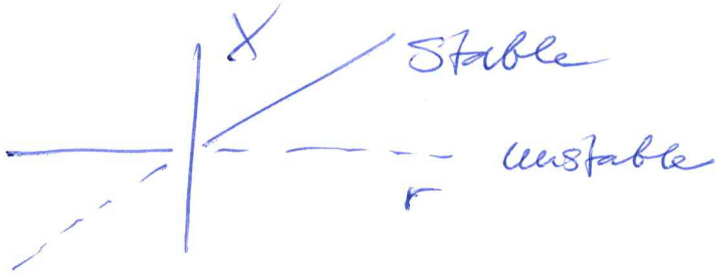


This in generic:

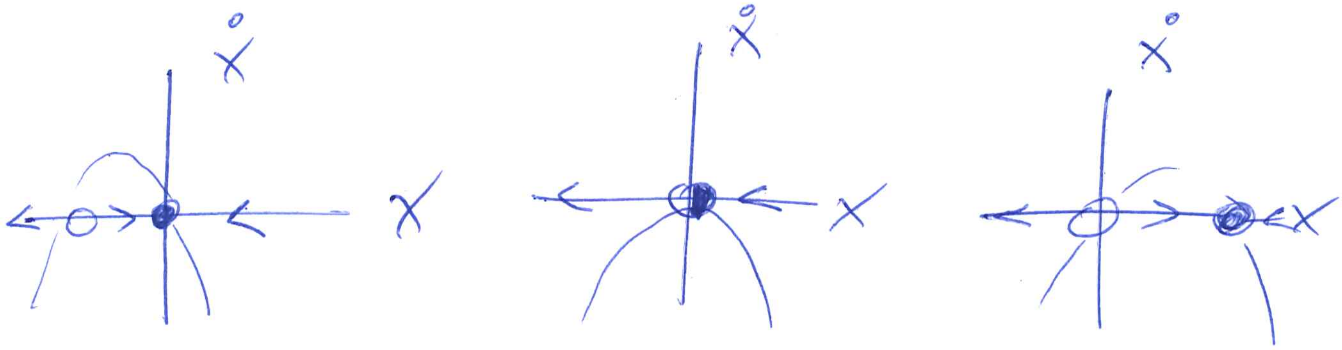
$$\dot{x} = r + x^2$$

$$T_{bottleneck} \approx \int_{-\infty}^{\infty} \frac{dt}{r+x^2} = \frac{\pi}{\sqrt{r}} .$$

Transcritical bifurcation



normal form $\dot{x} = rx - x^2$.

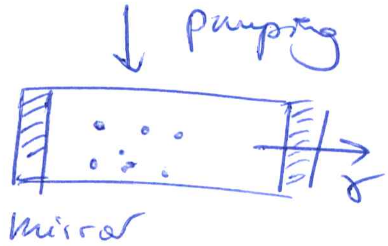


$r<0$ $\qquad$ $r=0$ $\qquad$ $r>0$

Minimal Model of solid-state lasers

$$\dot{n} = \underbrace{G n N}_{\text{gain}} - \underbrace{k n}_{\text{loss}}$$

stimulated emission



$$\dot{N} = - \underbrace{G n N}_{\substack{\text{stimulated}\\ \text{emission}}} - \underbrace{f N}_{\text{relaxation}} + \underbrace{p}_{\text{pumping}}.$$

(active system $\Rightarrow$ nonlinear dynamics)

$n \equiv$ number of photons

$N \equiv$ number of excited atoms.

Adiabatic elimination:

N relaxes much faster than n:

$\dot{N} = 0, \Rightarrow$

$$N = \frac{p}{f + Gn} \approx \underbrace{\frac{p}{f}}_{N_0} - \frac{pG}{f^2} n.$$

$$\dot{n} = G n \left(N_0 - \frac{pG}{f^2} n\right) - k n.$$

$\equiv$ transcritical bifurcation

Example:

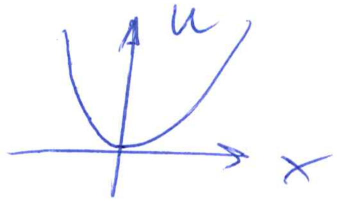
Double-well potential.

(proto-type of system with symmetries)

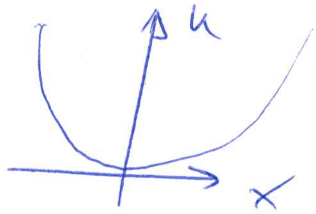
$$\frac{U}{U_0} = -\frac{1}{2} r x^2 + \frac{1}{4} x^4.$$

$$\tau \dot{x} = r x - x^3.$$

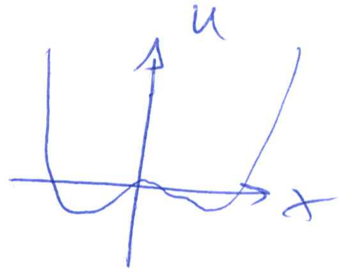
$$\tau = \gamma / U_0$$



$r<0$



$r=0$



$r>0$

critical slowing down

spontaneous symmetry breaking.

$$\tau \dot{x} = -x^3$$

(perturbations decay algebraically in time not exponentially)

Ising magnet

$$S_i = \pm 1 \,, \quad i = 1, \dots, N .$$

$$m = \langle S_i \rangle \equiv \text{magnetization}$$

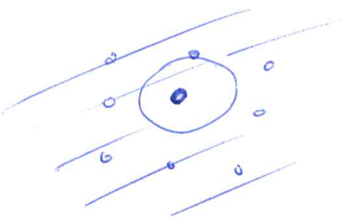
$m = 0$: paramagnet

$m > 0$: ferromagnet

Hamiltonian

$$H = \sum_{ij} J \, S_i S_j \,, \quad J \equiv \text{interaction strength}$$

Mean-field-approximation



$$\underbrace{\frac{H}{N}}_{\text{energy per spin}} \approx \cancel{N} J \cdot S_i \cdot n \cdot m$$

$n \equiv$ number of neighbors.

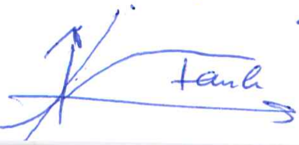
$$\underset{\text{partition sum.}}{Z} = \sum_{S_i = \pm 1} \exp\left(-\frac{H_i}{kT}\right).$$

$$\langle S_i \rangle = (+1) \, p_+ + (-1) \cdot p_-$$

$$= \frac{\exp\left(-\frac{J \cdot n \cdot m}{kT}\right) - \exp\left(+\frac{J \cdot n \cdot m}{kT}\right)}{Z}$$

$$= \tanh\left(\frac{J \cdot n \cdot m}{kT}\right).$$

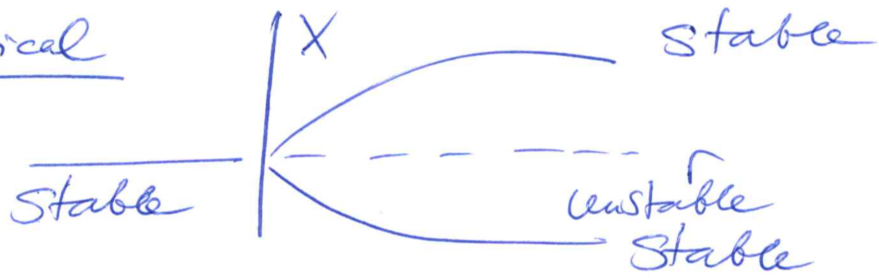
Self-consistency



$$\langle S_i \rangle \stackrel{!}{=} m \Rightarrow kT \cdot \tanh^{-1} m = J \cdot n \cdot m .$$

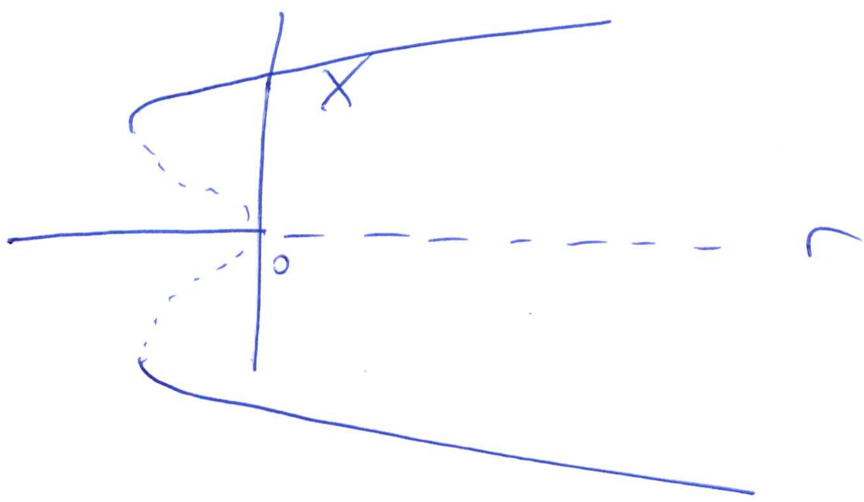
Pitchfork bifurcation

super-critical



$$\dot{x} = rx - x^3.$$

sub-critical



$$\dot{x} = rx + x^3 \underbrace{- x^5}.$$

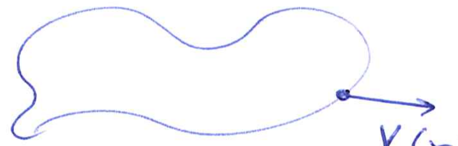
stabilizing nonlinearity to avoid blow-up

$\Rightarrow$ dangerous in engineering applications.

Index theory $\equiv$

topology of vector fields in phase plane

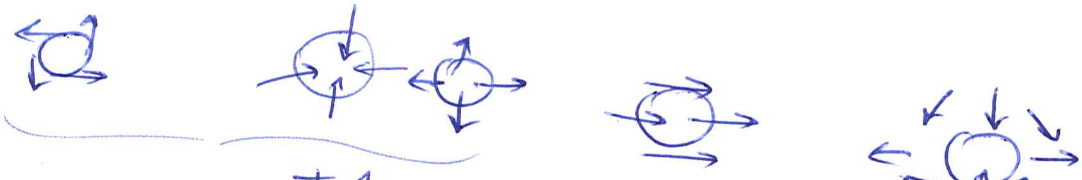
- for any closed curve C, define index I_C:



$I :=$ how often does $\underline{v}$ rotate when we go along C?

$= \frac{1}{2\pi} \oint \tan^{-1} \dot{y}/\dot{x}$

examples



+1   0   −1

- I_C depends continuously on C (as long as C does not pass through any fixed point, but continuous, integer-valued functions are constant!)

$\Rightarrow I_C = 0$ if C does not encircle any fixed point.

Proof: Shrink C to zero.

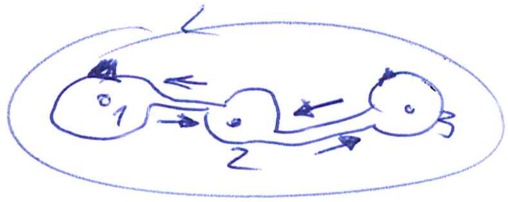
Theorem:

For each isolated fixed point x^*, we can define well-def'd index I_{x^*}. (take any curve encircling + only x^*).

If closed curve C encircles isolated fixed points $x_1^*, \dots, x_n^*$, then

$$I_C = I_1 + \dots + I_n.$$

Proof:



- Any closed orbit C has $I_C = +1$, hence must contain at least one fixed point.

More precisely

\# nodes + \# spirals + \# centers − \# saddles $= +1$

(for surfaces of higher topological genus depends on homotopy class of C)

Bifurcations revisited

Ic: continuous under change of bifurcation parameter, hence constant.

Saddle node bifurcation

Stable + unstable $\longrightarrow \emptyset$

● + ◐

1   −1   0.

Transcritical bifurcation

● + ○ $\longrightarrow$ ○ + ●

+1   +1   +1   +1

Pitchfork bifurcation

● $\longrightarrow$ ● + ◐ + ●

+1   +1   −1   +1