$\boxed{\text{Nonlinear dynamics.}}$

- Celestial mechanics

Newton

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial q}$$

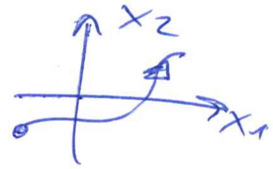
$$\dot{\underline{x}} = f(\underline{x})$$

- electric circuits
- control systems in engineering
- biological systems
- ecconomical models.

$\Rightarrow$ use a geometric way of thinking.

Basic concepts

- phase space $\underline{x}=(x_1,\dots,x_n)\in \mathbb{R}^n$
- trajectory $\underline{x}(t)$



- flow field $\dot{\underline{x}} = f(\underline{x})$

(no explicite time-dep: autonomous)

- fixed point

$$\underline{x}(t)=\underline{x}_0 \Leftrightarrow f(\underline{x}_0)=0.$$

- stability

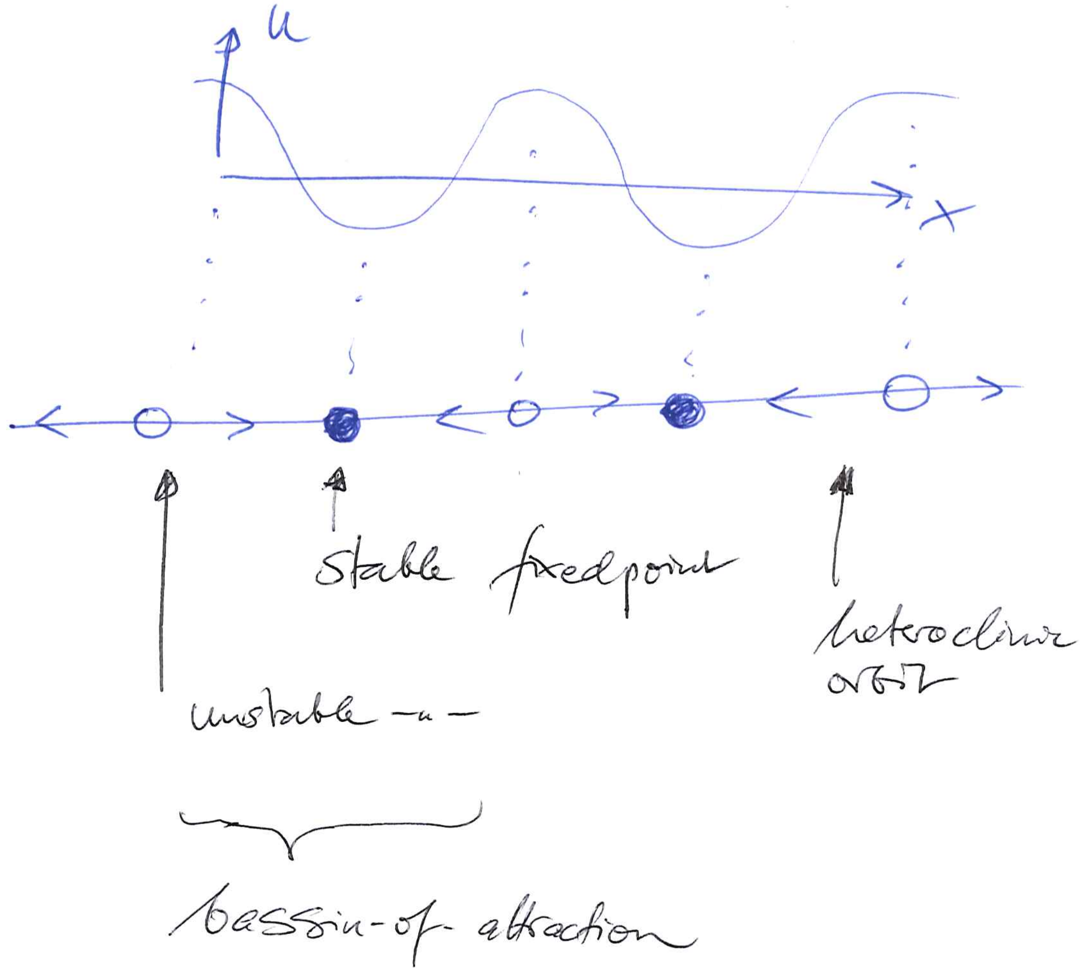
$\underline{x}_0 \equiv$ stable fixed point
iff
$\forall$ trajectories starting close to $\underline{x}_0$ converge to $\underline{x}_0$.

[We will use different definitions for stability later]

Example:

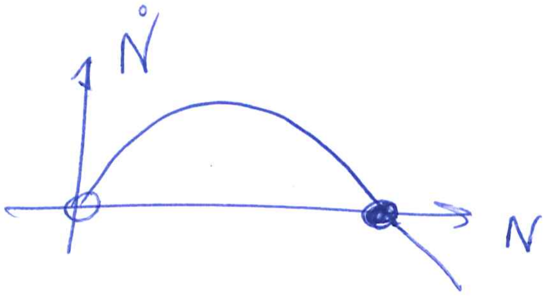
Overdamped motion in wash-board potential

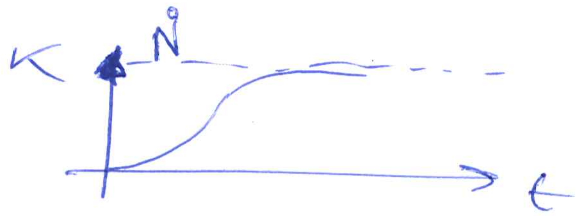
$$\gamma \dot{x} = -\frac{\partial U}{\partial x}, \quad U = U_0 \cos \lambda x.$$



Example Logistic growth

$$\dot{N} = r N \left(1 - \frac{N}{K}\right).$$





$\equiv$ Sigmoidal growth curve.

Existence and uniqueness.

$\Rightarrow$ trajectories in phase space cannot cross.

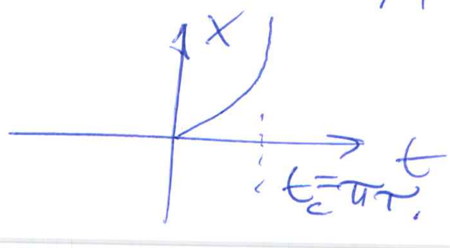
Blow-up of solutions

$$\tau \frac{dx}{dt} = \tau \dot{x} = 1 + x^2 \quad , \quad x(0) = 0.$$

$$\int_0^{x(t)} \frac{dx}{1+x^2} = \int_0^t \frac{dt}{\tau}$$

$$\tan^{-1} x = \frac{t}{\tau} \ (+c)$$

$$x = \tan t/\tau$$



Existence and uniqueness

(Picard-Lindelöf-theorem, Cauchy-Lipschitz-theorem)

$$\dot{\underline{x}} = f(\underline{x}), \quad \underline{x}(t_0) = \underline{x}_0 .$$

$f \equiv$ Lipschitz continuous, e.g. $|Df| < M < \infty$

$\Rightarrow$

there's unique solution for $[t_0, t_0+\varepsilon]$.

Idea of proof:

Construct iterative approximations

Way #1:

$$\underline{x}(n\Delta T) =: x_n$$

Euler scheme:

$$\underline{x}_n = \underline{x}_{n-1} + f(\underline{x}_{n-1})\Delta t .$$

$\Rightarrow$ let $\Delta t \to 0$.

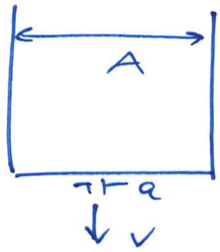
Way #2:

$$\mathcal{L} : \{\underline{x}(t)\} \mapsto \left\{\underline{x}_0 + \int_{t_0}^{t} \underline{f}(x(t))dt\right\}.$$

Show that $\|\mathcal{L}\| < 1$ on suitable function space (and small ε)

$\Rightarrow$ Banach fixed point theorem guarantees unique fixed point.

Example: Non-uniqueness of solutions.
"The empty bucket"



• mass conservation

$$A\dot{h} = -a\cdot v$$

We know:
$h(t=0)=0$.
When was
$h(t_0)=h_0$ for $t_0<0$?

• energy conservation

$$\frac{dm}{2}v^2 = g\cdot h\cdot dm$$

$$\Longrightarrow$$

$$v=\sqrt{2gh}$$

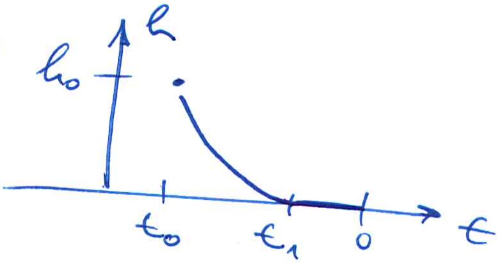
$$\frac{dh}{dt}=\dot{h}=-\frac{a}{A}\sqrt{2gh}$$

$$\int_{h_0}^{0}\frac{dh}{\sqrt{h}}=-\frac{a}{A}\sqrt{2g}\int_{t_0}^{t_1}dt$$

$$-2\sqrt{h}\Big|_{h_0}^{0}=-\frac{a}{A}\sqrt{2g}\,(t_1-t_0)$$

$$h = \;=\left(\frac{a}{A}\right)^2\frac{g}{2}(t_1-t_0)^2$$

$$\sqrt{\frac{2h_0}{g}}\cdot\frac{A}{a}=t_1-t_0$$



Celestial mechanics

2 bodies:

$$m_1 \ddot{\underline{r}}_1 = \underline{F}_{2,1}$$

$$m_2 \ddot{\underline{r}}_2 = \underline{F}_{1,2}, \quad \underline{F}_{2,1} = -\underline{F}_{1,2} = -\gamma \frac{m_1 m_2}{r^2} \hat{\underline{r}}$$

$$\Downarrow$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \qquad \underline{r} = \underline{r}_1 - \underline{r}_2$$

$$\mu \ddot{\underline{r}} = \underline{F}_{2,1} \quad \& \quad \text{center-of-mass}$$

$$\underline{R} = \frac{m_1}{m_1+m_2} \underline{r}_1 = \frac{m_2}{m_1+m_2} \underline{r}_2 = 0$$

(wlog)

1 body problem

3 body problem:

- would solve longitude problem [Newton].
- unstable [Poincaré]

- Invariants of motion:
  - center of mass $\underline{R}$
  - energy $\underline{E} = E_{kin} + E_{pot}$ = Hamiltonian system
  - angular momentum $\underline{L} = \mu r^2 \dot{\theta} \underline{k}$

$\Rightarrow$ motion in plane parallel to $\underline{L}$

How many degrees of freedom remain?

$$n=2: \quad 6n = 12$$

Noether theorem:

$$-6 \quad (\underline{R})$$

time-invariance $\quad -1 \quad (E)$

rotational invariance $\quad -3 \quad (\underline{L})$

$$= \underline{\underline{2}} \text{ DOF}: (r, \theta).$$

$$\dot\theta = \frac{L}{\mu r^2} \quad (\equiv \text{Kepler's } 2^{nd} \text{ law}).$$

$$E := E_{kin} + E_{pot}$$
$$= \frac{1}{2}\mu \dot r^2 + \frac{1}{2}\mu r^2 \dot\theta + \frac{\gamma m_1 m_2}{r}$$

$\Rightarrow$ Solve for $\dot r$

$$\Rightarrow \dot r = \sqrt{\frac{2}{\mu}} \left(E - \frac{1}{2}\frac{L^2}{\mu r^2} + \frac{\gamma m_1 m_2}{r}\right)^{1/2}$$

$\Rightarrow$ $\dot\theta / \dot r = d\theta/dr$ is seperable

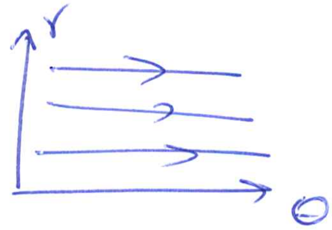
$\Rightarrow$ Kepler orbits

$$r = \frac{r_0}{1 - \varepsilon \cos\theta} \qquad r_0 = \frac{L}{\mu g m_1 m_2}$$

$$\varepsilon = \sqrt{1 + \frac{2EL}{\mu g^2}}$$

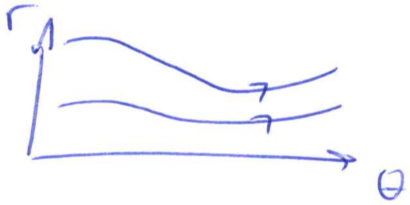
$\equiv$ eccentricity.

$\varepsilon = 0$: circle



phase-space foliated with neutrally stable periodic orbits.

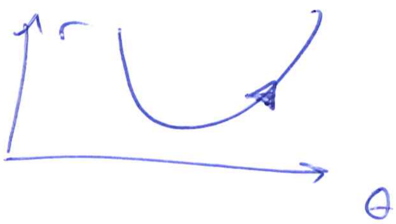
$0 < \varepsilon < 1$: ellipse



* $\varepsilon = 1$: parabola

bifurcation

$\varepsilon > 1$: hyperbola



escape to infinity