

Guaranteeing global synchronization in networks with stochastic interactions

Johannes Klinglmayr^{1,3}, Christoph Kirst², Christian Bettstetter¹ and Marc Timme²

¹ Institute of Networked and Embedded Systems, Lakeside Labs, University of Klagenfurt, 9020 Klagenfurt, Austria

² Network Dynamics Group, Max Planck Institute for Dynamics and Self-Organization, 37073 Göttingen, Germany

E-mail: johannes.klinglmayr@aau.at

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Abstract. We design the interactions between oscillators communicating via variably delayed pulse coupling to guarantee their synchronization on arbitrary network topologies. We identify a class of response functions and prove convergence to network-wide synchrony from arbitrary initial conditions. Synchrony is achieved if the pulse emission is unreliable or intentionally probabilistic. These results support the design of scalable, reliable and energy-efficient communication protocols for fully distributed synchronization as needed, e.g., in mobile phone networks, embedded systems, sensor networks and autonomously interacting swarm robots.

³ Author to whom any correspondence should be addressed.

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1. Introduction

Synchronization emerges in a variety of systems ranging from fireflies and neural networks in biology, to coupled lasers, wireless communication and Josephson junctions in physics and engineering [1–12]. Sometimes the goal of technological and medical systems is to avoid synchrony, e.g., during Parkinson tremor or epileptic seizures [4–7]. In many other systems, such as heart pacemakers [8], lasers [9], electric power grids [10] and communication technologies [11], synchronous dynamics of the units in a network is actually often intended. For instance, in the growing field of wireless embedded systems, a self-organizing approach to achieve synchrony seems to be a promising way of arranging slots and frames for data packet transmission without reference to a central unit [12–15]. Such self-organized dynamics should quickly adjust to changes and be scalable to large networks [12, 13].

The Kuramoto model of continuously coupled phase oscillators provides a simple, widely used paradigm for analyzing and designing synchronization [16]. Synchrony is indeed very common in such systems, but it is not always achieved, as it depends on network size and topology and also on the choice of initial conditions (cf [17]). Stochastic interactions may further hinder the synchronization process. Instead of time-continuous coupling, modern approaches to synchronization are often based on time-discrete pulse coupling. This approach appears to be more promising and fits better with the nature of packet-based communication used in wireless communication networks [12, 14, 15]. However, it remains an open problem how to reliably achieve network-wide synchrony in such a general setting.

To ensure proper functioning, synchrony is required to emerge across the entire network from arbitrary initial conditions, for arbitrary connected network topologies and for variably changing interaction delays. Moreover, it should be robust against stochastic communication errors. However, to our knowledge, so far there is no known pulse coupled oscillator system that satisfies all these requirements simultaneously. In particular, theoretical results suffer from constraints regarding at least one of the requirements: synchronization may be guaranteed only in the absence of delays [18–20], only for specific network topologies [21–25] or only for a certain subset of initial conditions [26–29].

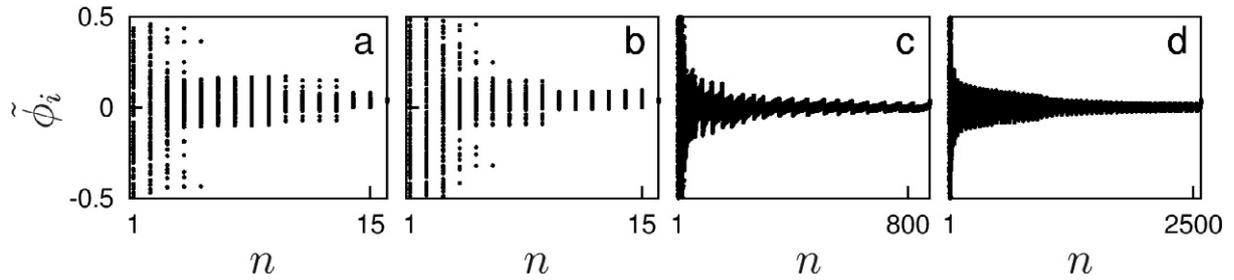


Figure 1. Synchronization on different network topologies. Centered phases ($\tilde{\phi}_i := (\phi_i + 0.5 \bmod 1) - 0.5$) at every pulse sending event at time t_n , $n \in \mathbb{N}$, starting from random initial conditions in different Watts–Strogatz graphs [30] ($N = 100$)⁴: (a) fully connected (rewiring prob. 0, degree 99), (b) random graph (rewiring prob. 1, degree 50), (c) small world (rewiring prob. 0.05, degree 6) and (d) circular grid (rewiring prob. 0, degree 6). Note the different time scales for synchronization.

In this paper, we design the pulse interactions of oscillator networks and explicitly exploit their stochastic features to guarantee global synchronization, simultaneously satisfying all the requirements listed above. The designed model class is based on pulse coupling functions that combine phase advancing and retarding with zero interaction (refractory period) close to the synchronized state. Moreover, we exploit unreliable (or intentionally stochastic) pulse communication [2, 3] as an essential feature to ensure synchronization from arbitrary initial conditions. In this setting synchronization emerges via two collective steps: firstly, oscillators arrange their phases to end up in an invariant subset of the state space. Secondly, within this subset they decrease their phase differences to finally achieve synchronization. Our key discovery is that a single pulse almost surely triggers a trackable chain of events that leads to synchrony. The proposed coupling thereby guarantees network-wide synchronization, which emerges for all (connected) network topologies in the presence of stochastic interactions from arbitrary initial conditions (cf figure 1).

2. Designing stochastic pulse interactions

Consider a connected undirected network of N oscillators $i \in I = \{1, \dots, N\}$ described by a scalar phase $\phi_i \in [0, 1]$ that evolves freely via

$$\frac{d}{dt}\phi_i = 1. \quad (1)$$

Whenever an oscillator's phase passes threshold 1, it resets its phase to 0, i.e.

$$\phi_i(t) = 1 \quad \Rightarrow \quad \phi_i(t^+) = \lim_{s \searrow 0} \phi_i(t+s) = 0 \quad (2)$$

⁴ We used uniformly distributed random initial conditions, the delay τ_{ij}^n is uniformly distributed in $[0, 0.02]$. Simulation is regarded as synchronized when bound $d_I \leq 0.02$ is reached, timeout at 2000 cycles, update function as in (5) with $h_1(\phi) = 0.2458\phi + 0.0151$ and $h_2(\phi) = 0.276\phi + 0.724$.

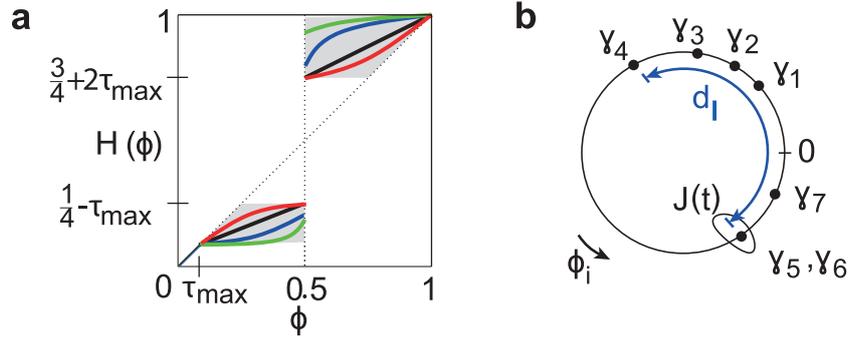


Figure 2. Model and definitions. (a) Pulse response function $H(\cdot)$, see (5). We consider any strictly monotonically increasing function in the gray-shaded areas. (b) Snapshot of the oscillators on the phase circle at time t . Indicated is the permutation γ_i of the indices that yields the ordering of the oscillators, the diameter d_I as defined in (7) and the lower boundary set $J(t) = \{\gamma_5, \gamma_6\}$; see (12). Note $J(t)$ can have several elements.

and emits a pulse with probability

$$0 < p_{\text{send}} < 1. \quad (3)$$

The oscillator is then called to fire. In technical systems such stochastic interactions are present due to noise and fading [31]. The time of the n th fire event in the network is denoted by t_n . Each pulse emitted by oscillator i is received by the neighboring oscillators $j \in \mathcal{N}_i$ after a stochastic delay $\tau_{ij}^n \in [0, \tau_{\text{max}}]$. Technical systems experience such delays due to varying processing times and channel access delays [31]. We assume that delays arbitrarily close to the lower bound exist. Upon reception of a pulse, a receiving oscillator j adjusts its phase according to

$$\phi_j((t_n + \tau_{ij}^n)^+) = H(\phi_j(t_n + \tau_{ij}^n)). \quad (4)$$

The pulse response function is defined via

$$H(\phi) = \begin{cases} \phi & \phi \leq \tau_{\text{max}} & \text{refractory,} \\ h_1(\phi) & \tau_{\text{max}} < \phi \leq \frac{1}{2} & \text{inhibitory,} \\ h_2(\phi) & \frac{1}{2} < \phi \leq 1 & \text{excitatory,} \end{cases} \quad (5)$$

where h_1 and h_2 are smooth functions satisfying $\frac{dh_1}{d\phi}, \frac{dh_2}{d\phi} > 0$; $h_1(\tau_{\text{max}}) = \tau_{\text{max}}$, $h_1(\frac{1}{2}) \leq \frac{1}{4} - \tau_{\text{max}}$; and $h_2(\frac{1}{2}^+) \geq \frac{3}{4} + 2\tau_{\text{max}}$, $h_2(1) = 1$, $h_2(\phi) \neq \phi$, for all $\phi \in (\frac{1}{2}, 1)$ (cf figure 2(a)). Consistency of these conditions requires a maximum possible delay of $\tau_{\text{max}} \leq \frac{1}{8}$. The phase of a receiving oscillator is retarded if $\phi \in (\tau_{\text{max}}, \frac{1}{2}]$ and advanced if $\phi \in (\frac{1}{2}, 1)$. If $\phi \in [0, \tau_{\text{max}}]$, it is in the refractory part of $H(\cdot)$. As is common for such systems (and motivated by real-world networks) we assume that the system is started without pulses in transmission such that at all later times, all pulses are generated by the system itself.

We define the distance of two phases by

$$d(\phi_i, \phi_j) := \min(|\phi_i - \phi_j|, 1 - |\phi_i - \phi_j|), \quad (6)$$

and set $d_{ij}(t) = d(\phi_i(t), \phi_j(t))$. We define the diameter of I as the shortest phase interval that contains all oscillators in I (cf figure 2(b)), i.e.

$$d_I := 1 - \max \left(\{\phi_{\gamma_{i+1}} - \phi_{\gamma_i}\}_{i=1}^{N-1} \cup \{\phi_{\gamma_1} + 1 - \phi_{\gamma_N}\} \right), \quad (7)$$

where γ_i is the permutation of indices γ_i , $i = 1, \dots, N$, such that $\phi_{\gamma_i} \leq \phi_{\gamma_{i+1}}$. In the remainder of this paper, we show that the class of pulse coupling functions (5) with stochastic pulse interactions (3) and stochastic delays guarantees network-wide synchrony from arbitrary initial conditions and for arbitrary (connected) networks.

3. Synchronization is guaranteed

The synchronization process has two qualitatively different stages: after the system has reached an invariant subset of state space, all oscillators synchronize their phases within that subspace.

3.1. Synchronization within an invariant subset

We start by showing that there is a subset in state space that is invariant under the stochastic dynamics and that almost surely all states in this subset reach the fully synchronized state asymptotically. We define this subset of state space to contain all states that satisfy $d_I \leq \frac{1}{2} - \tau_{\max}$. In the following, we show that this is an invariant set.

Assume that the system has reached a state in this set at time t_* , i.e.

$$d_I(t_*) \leq \frac{1}{2} - \tau_{\max}. \quad (8)$$

Then for all $(j, k) \in I^2$ with $d_{jk}(t_*) = d_I(t_*)$ and all $i \in I$, we have (cf also figure 2(b))

$$d_I(t_*) = d_{ji}(t_*) + d_{ik}(t_*). \quad (9)$$

The phases evolve uniformly (1) until either a pulse is emitted or received at t_e ; thus $d_I(t_e) = d_I(t_*)$. If a pulse is emitted, equations (2) and (6) give $d_I(t_e^+) = d_I(t_*)$. If a pulse, generated by oscillator i at time t_n , is received by oscillator j at $t_e = t_n + \tau_{ij}^n$, we observe $\phi_j(t_e^+) = \phi_i(t_e) \in [0, \tau_{\max}]$ due to (1) and the refractory part of $H(\cdot)$ (5). We consider $d_{ij}(t_e^+) = d(\phi_i(t_e), H(\phi_j(t_e)))$ for three different cases (cf also figure 2(a)):

- (i) for $\phi_j(t_e) \in [0, \tau_{\max}]$, we have: $H(\phi_j(t_e)) = \phi_j(t_e) \Rightarrow d_{ij}(t_e^+) = d_{ij}(t_e)$.
- (ii) for $\phi_j(t_e) \in (\tau_{\max}, \frac{1}{2}]$, we have: $\tau_{\max} < H(\phi_j(t_e)) < \phi_j(t_e) \Rightarrow d_{ij}(t_e^+) < d_{ij}(t_e)$.
- (iii) for $\phi_j(t_e) \in (\frac{1}{2}, 1)$, we have: $1 > H(\phi_j(t_e)) > \phi_j(t_e) \Rightarrow d_{ij}(t_e^+) < d_{ij}(t_e)$.

In total, the phase advancing, retarding and refractory parts of the interaction function $H(\cdot)$ ensure that d_{ij} does not increase. As all other phases do not change, we conclude that for all $k \in I$, $d_{ik}(t_e^+) \leq d_{ik}(t_e)$. Thus via (9) we conclude that $d_I(t_e^+) \leq d_I(t_*)$. Repeating this argument inductively for the next events, it follows that if (8) holds

$$d_I(t) \leq d_I(t_*) \quad \text{for all } t \geq t_*. \quad (10)$$

We now show that the diameter d_I of all states in the invariant subset, defined via (8), decreases to zero almost surely, i.e.

$$\mathbb{P} \left[\lim_{t \rightarrow \infty} d_I(t) > 0 \mid d_I(t_*) \leq \frac{1}{2} - \tau_{\max} \right] = 0. \quad (11)$$

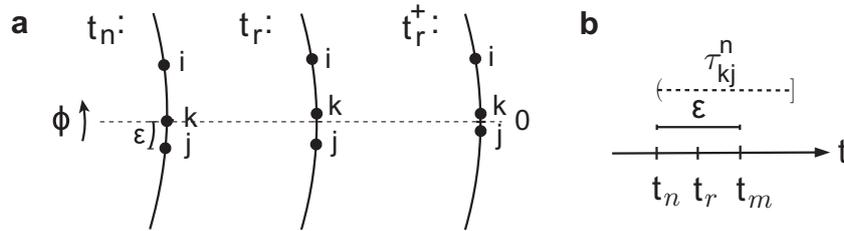


Figure 3. Phase distances decrease if not synchronized. (a) Zoom on the phase circle around 0. In this example, oscillator i is leading at t_n , oscillator j is last and $k \in \mathcal{N}_j$ is neighboring j . Oscillator k emits a pulse that oscillator j receives at t_r , adjusting its phase at t_r^+ . (b) Example of an event sequence that leads to a diameter change. At t_n oscillator k emits a pulse with an arrival time $t_r \in (t_n, t_n + \tau_{\max}]$ at oscillator j . If $\tau_{kj}^n \in (0, \epsilon)$ the pulse is received before oscillator j fires at t_m and its phase is advanced.

The proof is as follows: if the diameter repeatedly decreases, (11) follows. Thus, we assume for the rest of the proof that there is a time t_K such that for all $t \geq t_K$ we have $d_I(t) = d_I(t_K) =: c > 0$ and show that this leads to a contradiction. Therefore, we consider the lower boundary set (cf also figure 2(b))

$$J(t) := \{j \in I : \exists i \in I \text{ s.t. } \phi_i(t) - \phi_j(t) = d_I(t) \vee \phi_i(t) + 1 - \phi_j(t) = d_I(t)\}. \quad (12)$$

We show that—under the assumption—it first does not gain elements, i.e. for any oscillator j we have: if $j \notin J$ at $t \geq t_K$, then $j \notin J$ at any $t' > t$ (see appendix A.1 for technical details). Secondly, J almost surely loses elements: any oscillator repeatedly emits signals (appendix A.2). Hence, any oscillator j in J repeatedly receives signals, and since $d_I = c > 0$ and the delays have positive probability to be arbitrarily small, there is a positive probability that ϕ_j is not in the refractory part of $H(\cdot)$. Therefore, oscillator j has positive probability to drop out of J (appendix A.3); see figure 3 for illustration. Repeating these two arguments we find that almost surely there is a $t' \geq t_K$ with $J(t') = \emptyset$, which is a contradiction to (12).

Consequently, for every positive diameter with (8), the lower boundary set cannot increase and has a positive probability to decrease. Thus, no fixed point with $d_I > 0$ exists and together with (10) this yields (11).

3.2. The invariant subset is absorbing

In a second step, we show that for arbitrary initial conditions the system almost surely reaches a state in the invariant subset considered above. In particular, we prove that there is a time t_* (with probability 1) such that (8) and thus also (11) holds.

To this end, we start with an empty set S and form a chain of events that lets S absorb oscillators while d_S fulfills (8). We finally show that $S = I$ holds with positive probability. As the argument is independent of the phase positions of the oscillators and only depends on a firing pattern, every emitted pulse has a positive probability of starting this absorption process.

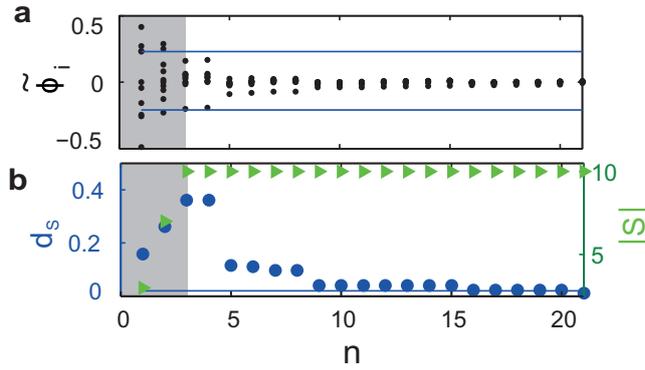


Figure 4. Synchronization process. (a) Phases ($\tilde{\phi}_i := (\phi_i + 0.5 \bmod 1) - 0.5$) at event times t_n for random initial conditions on a random network ($N = 10$, rewiring prob. 1, degree 5) (see footnote 4). The dotted lines indicate a phase difference of less than $\frac{1}{2} - \tau_{\max}$. All oscillators between these lines are combined in the set S . (b) The corresponding d_S and $|S|$ at event times t_n . Within the gray-shaded area: $|S| < N$; hence d_S can increase. As soon as $|S| = N$, i.e. $S = I$, d_I decreases (cf section 3.1).

Hence, as $t \rightarrow \infty$, the probability for (8) not to hold is zero and

$$\mathbb{P}[t_* < \infty] = 1. \quad (13)$$

See appendix B for details of the proof of (13).

Combining our statements, (13) guarantees that for an arbitrary connected network, there is a point in time t_* such that the condition (8) is fulfilled and the system has reached the invariant set. Consequently, the convergence guarantee (11) ensures global synchrony which yields (14), as phrased in our main result:

Theorem 1. *Arbitrary connected networks of pulse coupled oscillators of the form (1)–(5) started from arbitrary initial conditions synchronize almost surely, i.e.*

$$\mathbb{P}\left[\lim_{t \rightarrow \infty} \max_{i,j \in I} d_{ij}(t) = 0\right] = 1. \quad (14)$$

The two-step synchronization process is visualized in figure 4.

4. Relevance of stochastic pulses

To derive our main result (14), which shows guaranteed network-wide synchronization of oscillators from arbitrary initial conditions and independent of the network topology, we explicitly designed the pulse interactions (3)–(5).

In particular, the unreliable stochastic nature of the pulse transmission, $p_{\text{send}} < 1$, plays a constructive key role in the synchronization process, lacking which synchronization would be impossible for certain networks. For instance, consider a star graph (cf figure 5(c)) with $N > 4$, $\tau_{\max} < \frac{1}{8}$ and deterministic pulse emission $p_{\text{send}} = 1$. Assume the initial phases of the outer oscillators to be (roughly) equally spaced in $[0, 1]$ and the central node 1 having $\phi_1(0) = 0$. Every pulse from a non-central node then decreases ϕ_1 to $H(\phi_1) \leq \frac{1}{4} - \tau_{\max}$ as the pulse is always received before $\phi_1 > \frac{1}{2}$. Hence, the central node will never emit any pulse, the

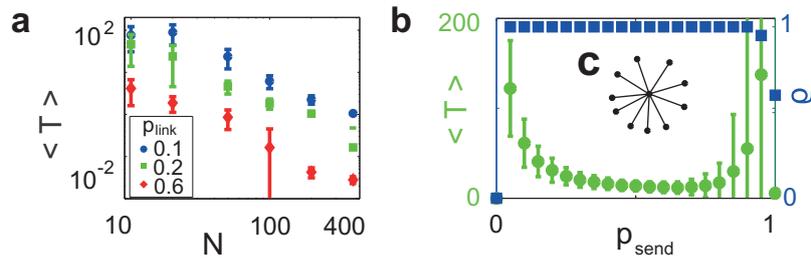


Figure 5. Rapid convergence. The mean and standard deviation of the convergence time T to synchrony (see footnote 4) depending on (a) the network size N in the Erdős–Rényi graphs (connection probability p_{link} and $p_{\text{send}} = 0.5$) and (b) the emission probability p_{send} in a star graph ($N = 11$, inset (c)). Additionally, we show the fraction ρ of randomly sampled initial conditions that synchronized (see footnote 4). As $p_{\text{send}} \rightarrow 1$ the synchronization time increases. For finite simulation time the numerically determined ρ drops already for p_{send} close to 1. For deterministic pulse emission $p_{\text{send}} = 1$, some initial conditions do not synchronize at all.

outer oscillators stay effectively uncoupled and thus synchronization cannot emerge. This is in contrast to the stochastic scenario where the outer oscillators are prevented from emitting a pulse at every cycle, giving the central node the possibility of firing (cf also (A.1)).

5. Time scales for synchronization and robustness

The theorem provides a synchronization guarantee but does not provide information on the time a system needs for convergence. In general, this synchronization time depends on the choice of $H(\cdot)$, τ_{max} and p_{send} , the underlying graph and the realization of the stochastic components. Numerical studies, however, demonstrate that synchronization is typically fast across systems (cf e.g. figures 1 and 4). Moreover, convergence time actually shrinks with network size (cf figure 5). For instance, figure 5(a) shows that convergence toward synchrony is accomplished within only a few cycles on the Erdős–Rényi random graphs. Remarkably, the convergence time and the number of emitted pulses per oscillator rapidly decrease with network size: in highly connected large networks, a few pulses are sufficient to elicit a cascade that aggregates all the phases within the invariant set (condition (8)) and thereby speeds up convergence. To give an example, for time scales typical in mobile phone networks [32], 100 devices could agree on common time slots required for communication on times as short as a few milliseconds. As user locations change on larger time scales, this consideration validates our approach with static graphs.

Additionally, we investigate how the randomness of pulse interactions affects synchronization time. We consider a star graph that is not guaranteed to synchronize for deterministic pulse emission, $p_{\text{send}} = 1$, and where synchronization heavily depends on p_{send} ; see figure 5(b). In fact, for $p_{\text{send}} = 1$, only 60% of the initial conditions reach full synchrony within 2000 cycles, while the others stay in asynchronous states akin to our counterexample above. Lowering p_{send} from unity, the fraction of initial states reaching full synchrony within that given cutoff time rapidly increases to one, while the synchronization time of those reaching synchrony rapidly

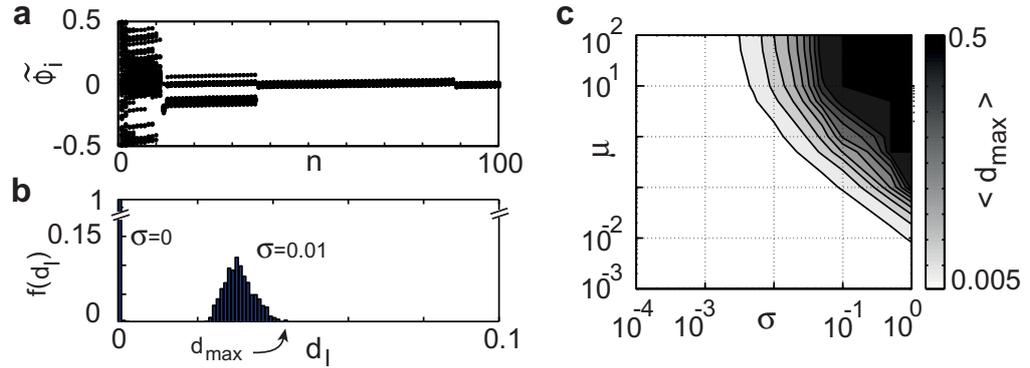


Figure 6. Robust global synchronization (on an Erdős–Rényi graph, $N = 100$, p_{link} and $p_{\text{send}} = 0.5$). (a) Phase evolutions for noisy phase rates (15) with $\sigma = 0.01$ and $\mu = 1$. The synchronization process is robust to frequency jitter. Due to noise, coinciding phases are replaced by an almost synchronous state with small diameter d_l . (b) The corresponding histogram of the d_l and its relative frequency $f(d_l)$ (evaluated over 100 cycles) in the steady state⁵ for $\sigma = 0$ and $\sigma = 0.01$, $\mu = 1$. (c) Maximum phase difference d_{\max} of the steady state averaged over 1000 realizations of a total of different Erdős–Rényi graphs ($N = 100$, p_{link} and $p_{\text{send}} = 0.5$), stochastic processes and initial conditions. For small noise levels the system synchronizes to low diameters $d_l < 0.01$ and is hence robust against phase and parameter noise.

drops. This suggests that relatively low pulse emission probabilities are sufficient to keep up the synchronization performance while at the same time the total number of transmitted pulses is decreased substantially. As pulse interactions consume energy in both the sender and the receiver, this is an advantageous property of our model concerning energy efficiency.

The synchronization process provides certain robustness effects and is not sensitive to phase perturbations. To study the robustness of the system we add noise to the intrinsic frequencies of the oscillators which captures both noise in phase and oscillator frequency. For technical real-world systems, drifts in frequencies usually occur. Therefore, we assume that instead of (1), the phase rates $\frac{d\phi_i(t)}{dt} = \omega_i(t)$ follow an Ornstein–Uhlenbeck process [33],

$$\dot{\omega}_i = \mu (1 - \omega_i) + \sigma \xi_i(t), \quad (15)$$

with independent white noise processes $\xi_i(t)$ obeying $\langle \xi_i(t), \xi_j(t') \rangle = \delta_{ij} \delta(t - t')$. These systems cannot maintain coinciding phases (cf figures 6(a) and (b)), but synchronize to low diameters for small σ and μ (cf figures 6(a)–(c))⁵.

We further remark that the probability that an evoked pulse decreases the phase spread increases with increasing d_l and thus further enhances robustness against noise and frequency drifts (see section 3).

⁵ We use a Euler approximation with time discretization $\Delta t = 10^{-3}$ [33]. As seen in figure 5(a), $\langle T \rangle$ is about 1 cycle for $N = 100$; correspondingly, we assume that a steady state is reached for $t \in [900, 1000]$ and define the maximum phase difference for a simulation run via $d_{\max} := \max_{t_n \in [900, 1000]} \max_{i, j \in I} d_{ij}(t_n)$.

6. Conclusions

In summary, we designed a stochastic pulse interaction for coupled oscillator networks to analytically guarantee that any such system almost surely synchronizes. This general result holds for any connected network, for arbitrary initial conditions and for any degree of randomness of the pulse emission, $0 < p_{\text{send}} < 1$. Our results do not depend on the details of the interaction function, but do rely on the coaction of the inhibitory, excitatory and refractory parts.

Early studies [34] (cf also [35]) on communication systems with delayed interactions found that full synchrony cannot (always) be achieved in the presence of distributed delays. At first glance, our result seems to contradict this finding. The discrepancy is resolved by noting that [34] allows signals that are always maximally delayed. We bypass this effect by both using a refractory period and assuming that arbitrarily small delays have positive probability.

Intriguingly, the stochastic nature of pulse emission, intuitively thought of as hindering synchrony (cf [2, 3]), provides two advantageous features simultaneously: it enables a synchronization guarantee and, at the same time, as fewer pulses are needed to be sent and received, it reduces operational communication efforts. From an application perspective, it nicely describes the nature of wireless communication networks where stochastic errors are omnipresent.

Complementary numerical studies (figure 5) indicate that synchrony is indeed achieved within a few cycles and emerges even faster for larger networks.

Synchronization is guaranteed for the assumptions discussed in this paper. A generalization of these assumptions is possible, such as for delays within an interval $[\tau_{\min}, \tau_{\max}]$, $\tau_{\min} > 0$, or directed networks. Assumptions on homogeneous phase rates and reoccurring delays arbitrarily close to the lower delay interval bound are vital for analytical guarantees. However, numerical simulation results show that this result is robust against parameter and phase noise.

In future studies, the guaranteed synchronization process described here may be applied to real world systems where synchronization is desired, ranging from pacemaker networks and neuron-like central pattern generators to interacting swarming robots and other wireless sensor networks [12, 14, 15, 36].

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Appendix A. Proofs of (11)

A.1. J does not gain elements

Firstly, an oscillator $k \notin J(t_r)$ can only become part of the lower boundary set during a reception event at time t_r . Then either the pulse has to force the phases of all oscillators in $J(t_r)$ above ϕ_k , or ϕ_k itself to the lower bound. Note that the pulse generating oscillator i has

$\phi_i(t_r) \in [0, \tau_{\max}]$. In the first situation we must have for all $j \in J(t_r)$, $\phi_j(t_r) \in (\frac{1}{2}, 1]$ and with (5), $\phi_j(t_r^+) > \phi_j(t_r)$. Thus, $d_I(t_r^+) < d_I(t_r)$, which is in contradiction to our assumption of constant $d_I(t)$. Thus, the only potential situation in which an oscillator k becomes a member of J is when $\phi_k(t_r) \in (\tau_{\max}, \frac{1}{2}]$. We then have for all $j \in J(t_r)$ $\phi_j(t_r) \in (1 - c, 1] \cup (0, \tau_{\max}]$ and $\phi_k(t_r^+) > \tau_{\max}$, since $\frac{dh_1}{d\phi} > 0$. Therefore, for all $j \in J(t_r^+)$ we have $d_{kj}(t_r^+) > 0$ and hence $k \notin J(t_r^+)$. In total this yields $J(t') \subset J(t_K)$ for all $t' \geq t_K$.

A.2. Every oscillator fires repeatedly

Due to the intrinsic rotation (1) oscillator i can be prevented from firing only if it receives pulses when $\phi_i \in (\tau_{\max}, \frac{1}{2}]$ from its neighbors \mathcal{N}_i to sufficiently retard its phase. As each pulse is emitted with probability $p_{\text{send}} < 1$, for the n th fire time of oscillator i , t_n^i , we have $\mathbb{P}[t_n^i = \infty] \leq (p_{\text{send}})^M$ for any $M \in \mathbb{N}$ and thus

$$\mathbb{P}[t_n^i < \infty] = 1. \quad (\text{A.1})$$

A.3. $|J|$ decreases with positive probability

Take $t_n > t_K$ and $k \in \mathcal{N}_{J(t_n)} := \cup_{j \in J(t_n)} \mathcal{N}_j$ with $\varepsilon := d_{kj}(t_n) > 0$ for all $j \in J(t_n)$ and $\phi_k(t_n) = 1$ (cf figure 3(a)). If $d_I(t_n) > 0$, such k and t_n exist almost surely as the network is connected and due to (A.1). Further, there is a positive probability that the pulse emitted by oscillator k at t_n is received by $j \in J(t_n) \cap \mathcal{N}_k$ in the next event at time $t_r \in (t_n, t_n + \varepsilon)$ (cf figure 3(b)). For this oscillator j we have $\phi_j \in [1 - c, 1)$ and thus $\phi_j(t_r^+) > \phi_j(t_r)$. By assumption, d_I is constant and since J does not gain elements (see appendix A.1) we must have $d_{jl}(t_r^+) > 0$ for all $l \in J(t_r^+)$, thus $j \notin J(t_r^+)$.

Appendix B. Proof of (13)

Take a subset $S \subset I$ of connected oscillators, define $d_S(t)$ analogous to (7) with $d_S(t_0) \leq \frac{1}{2} - \tau_{\max}$ at some time t_0 . Such an S always exists as $d_S = 0$ for $S = \{i\}$, $i \in I$. If $S = I$, we are done. Thus consider the case when the complement $S^c := I \setminus S$ is not empty. For any finite time interval there is a positive probability that all oscillators in S^c do not fire. Within such a time interval, we therefore treat S as an independent subset with no interactions between S^c and S . Then arguments similar to those that led to (11) ensure a positive probability that at some time $t' > t_0$ we have $d_S(t') \leq \tau_{\max}$. Next, with a positive probability an oscillator k of the edge set $\partial S := \{i \in S : S^c \cap \mathcal{N}_i \neq \emptyset\}$ fires at $t_n \geq t'$ and no other pulse is sent or received within the time interval $[t_n, t_n + \tau_{\max}]$. If at time $t_r \in [t_n, t_n + \tau_{\max}]$ an oscillator $i \in \mathcal{N}_k$ receives this pulse, the conditions on h_1 and h_2 , together with $\tau_{\max} \leq \frac{1}{8}$, ensure that if $\phi_i(t_r) \in [0, \frac{1}{2}]$, then $\phi_i(t_r^+) \leq \frac{1}{4} - \tau_{\max}$ and otherwise if $\phi_i(t_r) \in (\frac{1}{2}, 1)$, $\phi_i(t_r^+) \geq \frac{3}{4} + 2\tau_{\max}$. As $\phi_k(t_n + \tau_{\max}) \in [0, \tau_{\max}]$ this yields at time $t_n + \tau_{\max}^+$

$$d_{\mathcal{N}_k \cup \{k\}} \leq \frac{1}{2} - 2\tau_{\max}. \quad (\text{B.1})$$

Defining $S' := S \cup \mathcal{N}_k$ we obtain that

$$d_{S'} \leq d_S + d_{\mathcal{N}_k \cup \{k\}} \leq \frac{1}{2} - \tau_{\max}. \quad (\text{B.2})$$

Thus, now the larger set S' satisfies the conditions for the above argument and repeated applications thereof show that there is a positive probability that after a finite number of such steps we obtain an S with $S = I$. Hence, there is a positive probability that for any initial conditions after finitely many steps (8) holds. This yields (13).

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